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DISTRIBUTION LAWS IN WEAK POSITIONAL LOGICS[*]

A formal language is positional if it involves a positional connecitve, i.e. a connective of realization to relate formulas to points of a kind, like points of realization or points of relativization. The connective in focus in this paper is the connective "$\mathcal{R}$", first introduced by Jerzy Łoś. Formulas $\ulcorner \mathcal{R}_\alpha \varphi \urcorner$ involve a singular name α and a formula φ to the effect that φ is satisfied (true) relative to the position designated by α. In weak positional calculi no nested occurences of the connective "$\mathcal{R}$" are allowed. The distribution problem in weak positional +logics is actually the problem of distributivity of the connective "$\mathcal{R}$" over classical connectives, viz. the problem of relation between the occurences of classical connectives inside and outside the scope of the positional connective "$\mathcal{R}$".

CLASSICAL PROPOSITIONAL CALCULUS

For the sake of some later considerations it seems useful to most briefly specify some selected, commonly known concepts of the classical propositional calculus CPC. The alphabet of CPC contains a denumerable (infinite but enumerable) set $\mathbb{L}$ of schematic sentence letters. A sentence letter of CPC consists of the lower case letter "p" and any natural number in the lower index: "p_1", "p_2", "p_3" etc. are considered sentence letters of the classical

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propositional calculus in this paper. The connectives: "$\neg$" of negation, "$\wedge$" of conjunction, "$\vee$" of disjunction, "$\to$" of conditional and "$\equiv$" of equivalence are also to be used, together with parentheses, to form formulas. The formulas and the tautologies of the classical propositional calculus are to be defined in the standard way. The set of formulas of classical propositional calculus is the smallest collection containing all sentence letters as well as $\ulcorner(\neg\varphi)\urcorner$, for any formula φ, and $\ulcorner(\varphi \wedge \psi)\urcorner$, $\ulcorner(\varphi \vee \psi)\urcorner$, $\ulcorner(\varphi \to \psi)\urcorner$, $\ulcorner(\varphi \equiv \psi)\urcorner$, for any formulas φ, ψ. A valuation is any mapping from the set of sentence letters to the set of two truth values: truth and falsehood. Formulas are either true or false, but not both, under valuations, hence a formula is true if it is not false, false if it is not true, and converesly relative to every valuation. A sentence letter is true under a valuation if and only if it is assigen the value of truth under the valuation. A formula $\ulcorner(\neg\varphi)\urcorner$ is true under a valuation if and only if the formula φ is false under the valuation. A formula $\ulcorner(\varphi \wedge \psi)\urcorner$ is true under a valuation if and only if both formulas φ, ψ are true under the valuation. A formula $\ulcorner(\varphi \vee \psi)\urcorner$ is true under a valuation if and only if at least one of the formulas φ, ψ is true under the valuation. A formula $\ulcorner(\varphi \to \psi)\urcorner$ is true under a valuation if and only if either the formula φ is false or the formula ψ is true under the valuation. A formula $\ulcorner(\varphi \equiv \psi)\urcorner$ is true under a valuation if and only if either both formulas φ, ψ are true or both formulas φ, ψ are false under the valuation. A formula is tautology of classical propositional calculus if and only if it is true under every valuation. The set of all well formed formulas of the classical propositional calculus is to be called $\mathbb{F}$ and the set of all tautologies of the calculus is to be called $\mathbb{T}$.

Having excluded any ambiguity, we traditionally allow to omit parentheses. In such cases the following order of connectives: "$\neg$", "$\wedge$", "$\vee$", "$\to$", "$\equiv$" is to be preserved.

EXPRESSIONS OF POSITIONAL LOGIC

The positional alphabet consists of (a) a denumerable (infinite but enumerable) set $\mathbb{SL}$ of schematic sentence letters, (b) a denumerable set $\mathbb{PL}$ of schematic positional letters (also known as schematic indicators), (c) the constant connectives: "$\mathcal{R}$" of realization, "$\neg$" of negation, "$\wedge$" of conjunction, "$\vee$" of disjunction, "$\to$" of conditional and "$\equiv$" of equivalence, as well as (d) parentheses, serving for punctuation signs. All the connectives, but "$\mathcal{R}$", are *propositional*, whereas the connective "$\mathcal{R}$" is *positional*. The indicators are usually to be understood as (proper) names of positions of a kind, however, on the purely formal level it is only required that the sets $\mathbb{SL}$ and $\mathbb{PL}$ are mutually exclusive, i.e. $\mathbb{SL} \cap \mathbb{PL} = \varnothing$. Typical sentence letters of the set $\mathbb{SL}$ are the lower case letters: "p", "q" and "r", and typical positional letters of the set $\mathbb{PL}$ are the lower case letters: "a", "b" and "c". Nota bene the difference between the sets $\mathbb{SL}$ and $\mathbb{L}$. The set $\mathbb{QF}$ of quasi-formulas is characterized as the smallest collection, containing the set $\mathbb{SL}$ of schematic sentence letters, and closed under the following operations: $\ulcorner(\neg\varphi)\urcorner \in \mathbb{QF}$, provided $\varphi \in \mathbb{QF}$, and $\ulcorner(\varphi \wedge \psi)\urcorner$, $\ulcorner(\varphi \vee \psi)\urcorner$, $\ulcorner(\varphi \rightarrow \psi)\urcorner$, $\ulcorner(\varphi \equiv \psi)\urcorner \in \mathbb{QF}$, provided $\varphi, \psi \in \mathbb{QF}$. The set $\mathbb{AF}$ of atomic formulas contains exactly all sign clusters $\ulcorner(\mathcal{R}_\alpha\varphi)\urcorner$, in which $\alpha \in \mathbb{PL}$ and $\varphi \in \mathbb{QF}$. The atomic formula is to be read generally: at the point α it is the case that φ — or similarly. The set $\mathbb{FM}$ of all well formed formulas is characterized as the smallest collection containing the set $\mathbb{AF}$ of all atomic formulas and closed under the following operations: $\ulcorner(\neg\varphi)\urcorner \in \mathbb{FM}$, provided $\varphi \in \mathbb{FM}$, and $\ulcorner(\varphi \wedge \psi)\urcorner$, $\ulcorner(\varphi \vee \psi)\urcorner$, $\ulcorner(\varphi \rightarrow \psi)\urcorner$, $\ulcorner(\varphi \equiv \psi)\urcorner \in \mathbb{FM}$, provided $\varphi, \psi \in \mathbb{FM}$.

Having excluded any ambiguity, we traditionally allow to omit parentheses. In such cases the following order of connectives: "$\mathcal{R}$", "$\neg$", "$\wedge$", "$\vee$", "$\rightarrow$", "$\equiv$" is to be preserved.

Notice that in the set $\mathbb{FM}$ all schematic letters appear always within the scope of the connective "$\mathcal{R}$" and that no nested occurences of the connecive "$\mathcal{R}$" are allowed. Thus, supposing "a", "b", "c" $\in \mathbb{PL}$ and "p", "q", "r" $\in \mathbb{SL}$, the sign cluster:

$$\mathcal{R}_a\neg p \wedge \mathcal{R}_a(q \equiv r) \rightarrow (\mathcal{R}_b(p \vee q) \rightarrow \neg\mathcal{R}_c(p \vee r))$$

is an example of a well formed formula, being a member of the set $\mathbb{FM}$, whereas the clusters:

$$p \rightarrow \mathcal{R}_a p, \quad \mathcal{R}_a(\mathcal{R}_b p \equiv \mathcal{R}_c p) \wedge \mathcal{R}_b\mathcal{R}_c(q \wedge r)$$

do not belong to the set $\mathbb{FM}$ and are no formulas whatsoever. Hence, unlike typical languages the set $\mathbb{FM}$ is not closed under all connectives, vis. $\phi \in \mathbb{FM}$ but $\ulcorner\mathcal{R}_\alpha\phi\urcorner \notin \mathbb{FM}$ for some $\alpha \in \mathbb{PL}, \phi \in \mathbb{QF}$.

Those features constitute *weak* positional calculi, in opposition to the simply positional calculi as they have been developed mostly by Nicolas Rescher (RESCHER & URQUHART 1971; RESCHER & GARSON 1971).

THE SYSTEM MR

The system MR has been presented and examined by Tomasz Jarmużek and Andrzej Pietruszczak (2004), further examined by Marcin Tkaczyk (2013), by Jarmużek and Tkaczyk (2015) and by Anna Maria Karczewska (2018). It has been originally axiomatized by Jarmużek and Pietraszak (2004) as follows. Let $\alpha \in \mathbb{PL}, \varphi, \psi \in \mathbb{QF}$. The set of axioms of the system MR is the smallest subset of $\mathbb{FM}$ containing all the formulas:

$$e(\varphi), \tag{1}$$

provided $\varphi \in \mathbb{T}$ and e: $\mathbb{F} \to \mathbb{FM}$ is any uniform sustitution of elements of $\mathbb{FM}$ for all the sentence letters of CPC, i.e. a conservative extension of a mapping from $\mathbb{L}$ to $\mathbb{FM}$,

$$\mathcal{R}_\alpha e(\varphi), \tag{2}$$

provided $\varphi \in \mathbb{T}$ and e: $\mathbb{F} \to \mathbb{QF}$ is any uniform sustitution of elements of $\mathbb{QF}$ for all the sentence letters of CPC, i.e. a conservative extension of a mapping from $\mathbb{L}$ to $\mathbb{QF}$, as well as the following formulas:

$$\mathcal{R}_\alpha \neg\varphi \equiv \neg\mathcal{R}_\alpha \psi, \tag{3}$$

$$\mathcal{R}_\alpha \varphi \wedge \mathcal{R}_\alpha \psi \to \mathcal{R}_\alpha(\varphi \wedge \psi). \tag{4}$$

The rule of *Modus Ponens*:

$$\frac{\begin{array}{l}\varphi \\ \varphi \to \psi,\end{array}}{\varphi,} \qquad \text{(MP)}$$

for all $\varphi, \psi \in \mathbb{FM}$, is the unique primitive rule of inference. Notice that formulas like

$$\mathcal{R}_a p \vee \neg\mathcal{R}_a p$$

belong to the axiom collection (1), whereas formulas like

$$\mathcal{R}_a(p \vee \neg p)$$

belong to the axiom collection (2). So, by means of the principle (1) all substitutions of tautologies of CPC, being formulas, are axioms, whereas the principle (2) is a version of modal Gödelian generalization.

The set of theorems of the system MR is the smallest collection containing all the axioms and closed under the rule of *Modus Ponens* (JARMUŻEK & PIETRUSZCZAK 2004, 149–150).

In earlier works of mine I presented alternative axiomatics of the system MR. Instead of the axiom collection (2) it is enough to accept classical rules of mutual interchange of the connectives: "$\neg$", "$\wedge$", "$\vee$", "$\rightarrow$" and "$\equiv$", and strengthen the axiom (2) to the equivalence:

$$\mathcal{R}_{\alpha}\varphi \wedge \mathcal{R}_{\alpha}\psi \equiv \mathcal{R}_{\alpha}(\varphi \wedge \psi),$$

and the system thus constructed is exactly equal to the original version of MR (TKACZYK 2009, 2013). This information may turn out to be of some use in analyses to come.

Anna Maria Karczewska has proven that the system MR is also maximal in some special sense derived from Post's idea of completeness. Let $\alpha \in \mathbb{PL}$. The set of α-formulas is the smallest collection containing the formula $\ulcorner\mathcal{R}_{\alpha}\varphi\urcorner$ for every $\varphi \in \mathbb{QF}$, as well as formulas $\ulcorner(\neg\varphi)\urcorner$, for any α-formula φ, and $\ulcorner(\varphi \wedge \psi)\urcorner$, $\ulcorner(\varphi \vee \psi)\urcorner$, $\ulcorner(\varphi \rightarrow \psi)\urcorner$, $\ulcorner(\varphi \equiv \psi)\urcorner$, for any α-formulas φ, ψ. Shortly, α formulas are formulas with only one positional letter α. As Karczewska has shown every weak positional calculus L is inconsistent if the following three conditions are satisfied: (a) L is structura (closed under substitutions), (b) the system MR is a subsystem of L, and (c) at least one α-formula is a theorem of L or at least one inference rule with all the premises and the conclusion being α-formulas is in L which would be absent in the system MR (KARCZEWSKA 2018, 202).

It is worth noting thata the maximality Karczewska observed is a counterpart of collapsing in typical modal logic. Unlike typical modal logics, weak positional calculi do not collapse into classical propositional calculus, but reach the final degree of analogy to boolean connectives. It shows the system MR is an exemplification of that very degree to classical propositional calculus, and so a kind of ending point of a range of calculi.

As it has been already mentioned, in the system MR the connective "$\mathcal{R}$" is distributive over all sentential connectives, i.e., for all $\alpha \in \mathbb{PL}, \varphi, \psi \in \mathbb{QF}$, the following *distributive laws* are provable in MR:

$$\mathcal{R}_{\alpha}\neg\varphi \equiv \neg\mathcal{R}_{\alpha}\varphi, \tag{3}$$

$$\mathcal{R}_{\alpha}(\varphi \wedge \psi) \equiv \mathcal{R}_{\alpha}\varphi \wedge \mathcal{R}_{\alpha}\psi, \tag{5}$$

$$\mathcal{R}_{\alpha}(\varphi \vee \psi) \equiv \mathcal{R}_{\alpha}\varphi \vee \mathcal{R}_{\alpha}\psi, \tag{6}$$

$$\mathcal{R}_\alpha(\varphi \to \psi) \equiv \mathcal{R}_\alpha\varphi \to \mathcal{R}_\alpha\psi, \tag{7}$$

$$\mathcal{R}_\alpha(\varphi \equiv \psi) \equiv (\mathcal{R}_\alpha\varphi \equiv \mathcal{R}_\alpha\psi). \tag{8}$$

Actually, the system MR is the weakest positional logic bearing the feature (the letter "M" in the name of the calculus stands for "minimal", whereas the letter "R" invokes the shape of the connective "$\mathcal{R}$"). The proofs have been delivered by Jarmużek and Pietruszczak (2004, 151–153). Jarmużek and Pietruszczak have also proved the adequacy (i.e. soundness and completeness) result for the system MR with respect to a very simple and elegant structure (JARMUŻEK and PIETRUSZCZAK 2004, 154–159). Another semantic structure adequate with respect to the system MR has been described by Jarmużek and Tkaczyk (2015).

OBJECTIVE

A question arises naturally, since the calculus MR is actually the weakest one, containing *all* the distributive laws, what (if any) interesting calculi there are, weaker even than MR, containing at most *some* of the distributive laws or *none*? To find the answer is the objective of this paper.

First, however, it should be briefly explained, what alleged weak positional calculus could be considered interesting — in the sense of the just posed question.

DISTRIBUTIVE LAWS

All the distributive laws (1)–(6) are equivalences, hence they are quite strong. It is relativey easy to separate any of them. A much more ambitious plan is to analyze all the component distributive laws separately. Let $\alpha \in \mathbb{PL}$ and $\varphi, \psi \in \mathbb{QF}$. There are two considerable distributive laws for negations:

$$\mathcal{R}_\alpha\neg\varphi \to \neg\mathcal{R}_\alpha\varphi, \tag{RA}$$

$$\neg\mathcal{R}_\alpha\varphi \to \mathcal{R}_\alpha\neg\varphi. \tag{RB}$$

Of course, they make an outer negation follow from the inner one or conversely. Three alleged distributive laws for conjunctions are to be considered:

$$\mathcal{R}_\alpha(\varphi \wedge \psi) \rightarrow \mathcal{R}_\alpha\varphi, \tag{RC}$$

$$\mathcal{R}_\alpha(\varphi \wedge \psi) \rightarrow \mathcal{R}_\alpha\psi, \tag{RD}$$

$$\mathcal{R}_\alpha\varphi \wedge \mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \wedge \psi). \tag{RE}$$

According to the schema (RC) the first conjunct and according to the schema (RD) the second one follow from an inner conjunction. According to the schema (RE) an outer conjunction entails the inner one. Another three possible laws concern disjonctions:

$$\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha(\varphi \vee \psi), \tag{RF}$$

$$\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \vee \psi), \tag{RG}$$

$$\mathcal{R}_\alpha(\varphi \vee \psi) \rightarrow \mathcal{R}_\alpha\varphi \vee \mathcal{R}_\alpha\psi. \tag{RH}$$

According to the schema (RF) the first disjunct and according to the schema (RG) the second one entails an inner disjunction. According to the schema (RH) the outer disjunction follows from an inner one. Another three possible laws concern conditionals:

$$\neg\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha(\varphi \rightarrow \psi), \tag{RI}$$

$$\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \rightarrow \psi), \tag{RJ}$$

$$\mathcal{R}_\alpha(\varphi \rightarrow \psi) \rightarrow (\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha\psi). \tag{RK}$$

They are quite analogical to the schemata concerning disjunctions. According to the schema (RF) the outer negation of the antecedent and according to the schema (RG) the consequent entails an inner conditional. According to the schema (RH) the outer conditional follows from an inner one. Finally four alleged distributive laws concerning equivalences will be taken into consideration:

$$\mathcal{R}_\alpha\varphi \wedge \mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \equiv \psi), \tag{RM}$$

$$\neg\mathcal{R}_\alpha\varphi \wedge \neg\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \equiv \psi), \tag{RN}$$

$$\mathcal{R}_\alpha(\varphi \equiv \psi) \rightarrow (\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha\psi), \tag{RP}$$

$$\mathcal{R}_\alpha(\varphi \equiv \psi) \rightarrow (\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha\varphi). \tag{RQ}$$

The schemata (RM) and (RN) together make the inner equivalence follow from an outer one — both schemata together are equivalent to the schema:

$$(\mathcal{R}_\alpha\varphi \equiv \mathcal{R}_\alpha\psi) \rightarrow \mathcal{R}_\alpha(\varphi \equiv \psi).$$

However, for the sake of some applications it seems more comfortable to consider them separately. It seems also more useful to consider the schemata (RM) and (RN) rather than the schemata:

$$(\mathcal{R}_\alpha\varphi \to \mathcal{R}_\alpha\psi) \to \mathcal{R}_\alpha(\varphi \equiv \psi),$$
$$(\mathcal{R}_\alpha\psi \to \mathcal{R}_\alpha\varphi) \to \mathcal{R}_\alpha(\varphi \equiv \psi).$$

The antecedents of the above schemata are so weak, that the inner equivalence in question turns out much closer to an outer disjunction than to an outer equivalence. A single schema:

$$(\mathcal{R}_\alpha\varphi \to \mathcal{R}_\alpha\psi) \wedge (\mathcal{R}_\alpha\psi \to \mathcal{R}_\alpha\varphi) \to \mathcal{R}_\alpha(\varphi \equiv \psi)$$

would be yet more interesting. Nevertheless, the tools developed in this paper allow to easily and even immediately modify collections of distributive laws from one application to another. The schema (RP) make the right-side outer conditional and the schema (RQ) the left-side outer conditional follow from an inner equivalence.

Obviously, the schemata (RA) and (RB) are jointly equivalent to the schema (3), the schemata (RC)–(RE) are jointly equivalent to the schema (5), the schemata (RF)–(RH) are jointly equivalent to the schema (6), the schemata (RI)–(RK) are jointly equivalent to the schema (7) and the schemata (RM)–(RQ) are jointly equivalent to the schema (8).

WEAK POSITIONAL CALCULI

The weak positional calculi being considered in this paper spring into existence out of the axiom collection (1), the derivation rule (MP) and any selection of axiom schemata (RA)–(RQ). The bottom calculus, the weakest one is based on the axioms (1) and the rule (MP) solely (furthermore, the rule turns out derivable). It is called Zero calculus. Other calculi may be described simply by listing the additional axiom schemata accepted for the calculus. For example,

EHKPQ

is the calculus based on the axioms: (1), (RE), (RH), (RK), (RP), (RQ) and the rule (MP). The top calculus,

ABCDEFGHIJKMNPQ (A–Q for short),

is deductively equivalent to the calculus MR from Jarmużek and Pietruszczak (TKACZYK 2009; 2013). It turns out then that all the schemata (RA)–(RQ) are deductively perfectly independent, and hence separable from one another.

THEOREM 1: *A formula* $\varphi \in \mathbb{FM}$ *is provable in the system* Zero *if and only if* $\varphi = e(\psi)$ *for some tautology* ψ *of the classical propositional calculus and some uniform substitution* e *of members of* $\mathbb{FM}$ *for members of* $\mathbb{L}$.

The theorem 1 is an almost obvious corollary from the description of the system Zero. Every substitution of a tautology of the classical propositional calculus is an axiom of the collection (1) in the system Zero. Because the rule (MP) preserves being a tautology, no other formula is provable in the system Zero. Since the rule (MP) does not lead out of the set of tautologies of CPC and substitutions of all the tautologies belong to the axiom collection (1), it also get obvious that the rule (MP) is derivative in the system Zero . Actually an even slightly stronger theorem is provable.

THEOREM 2. *The system* Zero *is algebraically identical with* CPC.

PROOF. Notice that the cardinal number of each set: $\mathbb{L}$, $\mathbb{QF}$ and $\mathbb{PL}$ makes exactly $\aleph_0$. In the case of the sets $\mathbb{L}$ and $\mathbb{PL}$ the fact is simply presumed. In the case of the set $\mathbb{QF}$ it obviously follows from the fact that quasi-formulas are finite strings of elements of the infinite, denumerable set $\mathbb{SL}$ and the finite set of parentheses. It follows that the cardinal number of the carthesian product $\mathbb{PL} \times \mathbb{QF}$ also makes $\aleph_0$. Regard the connective "$\mathcal{R}$" just a letter and the pairs of positional letters and quasi-formulas indices to the letter. Of course, there is exactly $\aleph_0$ indices, hence, they may be easily numbered with natural numbers. Let $\mathtt{nr}(\alpha, \varphi)$ be the number of the pair $\langle \alpha, \varphi \rangle$, where $\alpha \in \mathbb{PL}, \varphi \in \mathbb{QF}$. There is an obvious one-to-one mapping from $\mathbb{L}$ onto $\mathbb{AF}$: the letter "$\mathcal{R}$", the indicator with α and the quasi-formula to be the unque counterpart to the letter "p" with the lower index $\mathtt{nr}(\alpha, \varphi)$ and conversely. For example, if it happens that $\mathtt{nr}(a, p \to q \vee r) = 48$, then the letter "$p_{48}$" is the counterpart of the letter-made formula "$\mathcal{R}_a(p \to q \vee r)$" and conversely. Since the atomic formulas in the system Zero are pefectly deductively independent, they may be equally regarded just sentence letters and the sets $\mathbb{L}$ and $\mathbb{AF}$ turn out to be perfectly interchangable from the algebraical point of view. Since outer connectives in all weak positional calculi are classical, the system Zero turns out to be actually the classical propositional calculus itself. **QED**

To understand the vital analogy, or even identity, between the system Zero and classical propositional calculus notice that the set $\mathbb{AF}$ of atomic formulas of a weak positional logic is denumerable, exactly like the set $\mathbb{L}$ of sentence letters of classical propositional calculus. Create any sequence of all members of the set $\mathbb{AF}$ and assign the letter $p_n \in \mathbb{L}$ to the n member of the sequence. Extending this maping over the connectives of classical propositional calculus one achieves a mapping from the set of theorems of the system Zero to the set $\mathbb{T}$ of tautologies of classical propositional calculus. Hence, every theorem of the system Zero is obviously a subsitution of a member of the set $\mathbb{T}$. It turns out members of the set $\mathbb{AF}$ in the system Zero work exactly like sentence letters in classical propositional calculus. They may by assigned truth values perfectly arbitrarily, like in the case of the set $\mathbb{L}$. And theorems are simply formulas true under every valuation. It seems therefore legitimately to say the system Zero *is* classical propositional calculus with its atomic formulas of the set $\mathbb{AF}$ being sentence letters. The system Zero with the feature just described and the system MR with Karczewska's sense of maximality constitute natural borders of a range of weak positional calculi of some kind. Those calculi are created vitally by posing distributivity conditions of the connective "$\mathcal{R}$" with respect to some, but not necessarily all, connectives.

THEOREM 3. *The system* A−Q *is deductively equivalent to the system* MR.

PROOF. Both calculi share the axiom collection (1) and the rule (MP). As all the distributive laws (3)–(8) are provable in the system MR, it contains the system A−Q. The axiom schema (3) is immediately achievable from the schemata (RA) and (RB), and the schema (4) is identical to the schema (RE). So, it is sufficient to prove all the axioms making the collection (2) are provable in the system A−Q. To do so, let $e(\varphi) = \ulcorner \mathcal{R}_\alpha \varphi \urcorner$, for any $\alpha \in \mathbb{PL}$ and $\varphi \in \mathbb{SL}$. Remember that all the distributive laws (3)–(8) are obviously derivable from the axiom schemata (RA)–(RQ). If $\psi \in \mathbb{T}$ is any tautology of the classical propositional calculus and $\alpha \in \mathbb{PL}$, the formula $\ulcorner \mathcal{R}_\alpha \psi \urcorner$ is provable in the system A−Q in the following way. Begin the proof with the axiom $e(\psi)$ of the collection (1) and use the distributive laws (3)–(8) to transform it equivalently into the formula $\ulcorner \mathcal{R}_\alpha \psi \urcorner$. **QED**

The domain of weak positional calculi has been thus outlined, where the system A-Q (i.e. MR) is the top and the system Zero (i.e. actually CPC) is the bottom one. The domain will be now investigated with respect to the distributive laws (RA)–(RQ).

MODEL TEMPLATE

The formal construction serving as semantics is so designed that all the propositional connectives outside the scope of the connective "$\mathcal{R}$" are perfectly classical, but inside its scope they may deviate with practically no limits. A model is any threesome

$$\mathfrak{M} = \langle \mathfrak{X}, \mathfrak{d}, \mathfrak{f} \rangle, \tag{9}$$

where

$$\mathfrak{X} \text{ is a non-empty set,} \tag{10}$$
$$\mathfrak{d} : \mathbb{PL} \to \mathfrak{X}, \tag{11}$$
$$\mathfrak{f} : \mathbb{QF} \to \wp(\mathfrak{X}). \tag{12}$$

Elements of the set $\mathfrak{X}$ are to be interpreted as points of relativization, $\mathfrak{d}(\alpha)$, for any $\alpha \in \mathbb{PL}$, is the unique element of $\mathfrak{X}$ denoted by the letter α, and $\mathfrak{f}(\varphi)$, for any $\varphi \in \mathbb{QF}$ is the subset of $\mathfrak{X}$ made exactly of the points relative to which φ is satisfied. The set $\mathfrak{X}$ may be canonically interpreted such that $\mathfrak{X} \subseteq \wp(\mathbb{QF})$, and the mapping $\mathfrak{f}$ may then simply mean conversed belonging to the effect that $x \in \mathfrak{f}(\varphi)$ if and only if $\varphi \in x$, for an $x \in \mathfrak{X}$..However, such interpretation is not necessary. The set $\mathfrak{X}$ may be a set of points in time, in space, space-time, a set of persons, possible worlds or in another way.

The most vital feature of the construction being presented is that $\mathfrak{f}$ is *any* mapping from $\mathbb{QF}$ to $\wp(\mathfrak{X})$, rather than an extension of a traditional mapping from $\mathbb{SL}$. That means, values taken on at compound quasi-formulas need not be uniquely determined by the values of their components. For example, mapping $x \in \mathfrak{f}(\varphi)$ and $x \in \mathfrak{f}(\psi)$ has no influence whatsoever on the possible mapping $x \in \mathfrak{f}(\varphi \wedge \psi)$. One simply maps quasi-formulas into subsets od the universe $\mathfrak{X}$. As it will shortly be shown, some extra constraints put on the mapping $\mathfrak{f}$ constitute models to definite calculi.

Formulas of weak positional language, i.e. elements of the set $\mathbb{FM}$, are true or false in a model. To be false in a model means exactly not to be true in it. So, for any model $\mathfrak{M}$ and any $\varphi \in \mathbb{FM}$, either $\mathfrak{M} \vDash \varphi$, or $\mathfrak{M} \nvDash \varphi$, but never both. Let $\alpha \in \mathbb{PL}$ and $\varphi \in \mathbb{QF}$:

$$\mathfrak{M} \vDash \ulcorner \mathcal{R}_\alpha \varphi \urcorner \text{ if and only if } \mathfrak{d}(\alpha) \in \mathfrak{f}(\varphi). \tag{13}$$

So, an atomic formula $\ulcorner \mathcal{R}_\alpha \varphi \urcorner$ is true in a model $\mathfrak{M}$ if and only if the point

$\mathfrak{d}(\alpha)$ belongs to the set $\mathfrak{f}(\varphi)$ in the model $\mathfrak{M}$. Truth conditions for compound formulas are classical:

$$\mathfrak{M} \vDash \ulcorner \neg\varphi \urcorner \text{ if and only if } \mathfrak{M} \nvDash \varphi, \tag{14}$$

$$\mathfrak{M} \vDash \ulcorner \varphi \wedge \psi \urcorner \text{ if and only if } \mathfrak{M} \vDash \varphi \text{ and } \mathfrak{M} \vDash \psi, \tag{15}$$

$$\mathfrak{M} \vDash \ulcorner \varphi \vee \psi \urcorner \text{ if and only if } \mathfrak{M} \vDash \varphi \text{ or } \mathfrak{M} \vDash \psi, \tag{16}$$

$$\mathfrak{M} \vDash \ulcorner \varphi \rightarrow \psi \urcorner \text{ if and only if } \mathfrak{M} \nvDash \varphi \text{ or } \mathfrak{M} \vDash \psi, \tag{17}$$

$$\mathfrak{M} \vDash \ulcorner \varphi \equiv \psi \urcorner \text{ if and only if } \mathfrak{M} \vDash \varphi, \psi \text{ or } \mathfrak{M} \nvDash \varphi, \psi, \tag{18}$$

for any $\varphi, \psi \in \mathbb{FM}$. Formulas may be considered as valid in a sense if and only if they are true in a set of models related to the sense.

SETS OF MODELS

The sets of models adequate for different calculi are achievable nearly algorithmically by imposing proper constraints on the mapping $\mathfrak{f}$. With respect to the connective of negation two constraints are considerable:

$$\mathfrak{f}(\neg\varphi) \subseteq -\mathfrak{f}(\varphi), \tag{19}$$

$$-\mathfrak{f}(\varphi) \subseteq \mathfrak{f}(\neg\varphi), \tag{20}$$

for any $\varphi \in \mathbb{QF}$. If imposed together, they make the connective of negation classical. With respect to the connective of conjunction three constraints are considerable:

$$\mathfrak{f}(\varphi \wedge \psi) \subseteq \mathfrak{f}(\varphi), \tag{21}$$

$$\mathfrak{f}(\varphi \wedge \psi) \subseteq \mathfrak{f}(\psi), \tag{22}$$

$$\mathfrak{f}(\varphi) \cap \mathfrak{f}(\psi) \subseteq \mathfrak{f}(\varphi \wedge \psi), \tag{23}$$

for any $\varphi, \psi \in \mathbb{QF}$. If imposed together, they make the connective of conjunction classical. With respect to the connective of disjunction also three constraints are considerable:

$$\mathfrak{f}(\varphi) \subseteq \mathfrak{f}(\varphi \vee \psi), \tag{24}$$

$$\mathfrak{f}(\psi) \subseteq \mathfrak{f}(\varphi \vee \psi), \tag{25}$$

$$\mathfrak{f}(\varphi \vee \psi) \subseteq \mathfrak{f}(\varphi) \cup \mathfrak{f}(\psi), \tag{26}$$

for any $\varphi, \psi \in \mathbb{QF}$. If imposed together, they make the connective of disjunction classical. With respect to the connective of conditional three constraints are considerable:

$$-\mathfrak{f}(\varphi) \subseteq \mathfrak{f}(\varphi \to \psi), \tag{27}$$
$$\mathfrak{f}(\psi) \subseteq \mathfrak{f}(\varphi \to \psi), \tag{28}$$
$$\mathfrak{f}(\varphi \to \psi) \cap \mathfrak{f}(\varphi) \subseteq \mathfrak{f}(\psi), \tag{29}$$

for any $\varphi, \psi \in \mathbb{QF}$. If imposed together, they make the connective of conditional classical. With respect to the connective of equivalence two constraints are considerable:

$$\mathfrak{f}(\varphi) \cap \mathfrak{f}(\psi) \subseteq \mathfrak{f}(\varphi \equiv \psi), \tag{30}$$
$$-\mathfrak{f}(\varphi) \cap -\mathfrak{f}(\psi) \subseteq \mathfrak{f}(\varphi \equiv \psi), \tag{31}$$
$$\mathfrak{f}(\varphi \equiv \psi) \cap \mathfrak{f}(\varphi) \subseteq \mathfrak{f}(\psi), \tag{32}$$
$$\mathfrak{f}(\varphi \equiv \psi) \cap \mathfrak{f}(\psi) \subseteq \mathfrak{f}(\varphi), \tag{33}$$

for any $\varphi, \psi \in \mathbb{QF}$. If imposed together, they make the connective of equivalence classical.

ADEQUATE MODELS

The model template (9) and the specific conditions (19)–(33) are so designed to deliver a simple algorithm of construction of adequate — i.e. both sound and complete — weak positional calculi from `Zero` to `MR`. The axiom collection (1) and the derivation rule (`MP`) are rigid, whereas all the schemata (`RA`)–(`RQ`) are flexible and may freely vary from one calculus to another. Any calculus thus constructed gets immediately an adequate semantics in the shape of a set of models of the template (9). The proofs I deliver are of course Lindenbaumian.

THEOREM 4. *For every weak positional calculus L and every $\varphi \in \mathbb{FM}$, if φ is provable in L, then φ is also true in every L-model $\mathfrak{M}$.*

PROOF. Due to the conditions (14)–(18) all the axioms of the collection (1) are true in any model of the template (9), furthermore the derivation rule (`MP`) preserves truth in any model of the kind. For every schema (`RA`)–(`RQ`), every example of the schema is true in any model meeting the correlated condition (19)–(33). So, all provable formulas are true in the models correlated to a calculus inquestion. **QED**

THEOREM 5. *For every weak positional calculus L, if Λ is any maximal consistent extension of L, then there exists such a discriminative L-model $\mathfrak{M}$ that, for every formula $\varphi \in \mathbb{FM}$, the formula φ is true in $\mathfrak{M}$ if and only if φ belongs to Λ.*

PROOF. Let Λ be any maximal consistent extension of a weak positional calculus L. Assume, for every $\alpha \in \mathbb{PL}, \varphi \in \mathbb{QF}$, that $\mathfrak{d}(\alpha) \in \mathfrak{f}(\varphi)$ if and only if $\ulcorner \mathcal{R}_\alpha \varphi \urcorner \in \Lambda$. If $\varphi \in \mathbb{SL}$, the value of $\mathfrak{f}$ at φ is determined by the definition of a maximal consistent set. Hence, suppose the theorem is satisfied for a subset Γ of Λ.

§ 1. Suppose $\ulcorner \mathcal{R}_\alpha \varphi \urcorner \in \Gamma$. If the schema (RA) is provable, then $\ulcorner \mathcal{R}_\alpha \neg\varphi \urcorner \notin \Lambda$ and the condition (19) is satisfied. Otherwise there exist both consistent extension of Γ containing $\ulcorner \mathcal{R}_\alpha \neg\varphi \urcorner$ and one non containing it.

§ 2. Suppose $\ulcorner \mathcal{R}_\alpha \varphi \urcorner \notin \Gamma$. If the schema (RB) is provable, then $\ulcorner \mathcal{R}_\alpha \neg\varphi \urcorner \in \Lambda$ and the condition (20) is satisfied. Otherwise there exist both consistent extension of Γ containing $\ulcorner \mathcal{R}_\alpha \neg\varphi \urcorner$ and one non containing it.

§ 3. Suppose $\ulcorner \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \mathcal{R}_\alpha \psi \urcorner \in \Gamma$. If the schema (RE) is provable, $\ulcorner \mathcal{R}_\alpha (\varphi \wedge \psi) \urcorner \in \Lambda$ and the condition (23) is satisfied. Otherwise some consistent extensions of Γ contain the formula inquestion, other do not.

§ 4. Suppose $\ulcorner \mathcal{R}_\alpha \varphi \urcorner \notin \Gamma$ or $\ulcorner \mathcal{R}_\alpha \psi \urcorner \notin \Gamma$. In the former case, if the schema (RC) is provable, $\ulcorner \mathcal{R}_\alpha (\varphi \wedge \psi) \urcorner \notin \Lambda$ and the condition (21) is satisfied. In the latter case, if the schema (RD) is provable, $\ulcorner \mathcal{R}_\alpha (\varphi \wedge \psi) \urcorner \notin \Lambda$ and the condition (22) is satisfied. If neither of schemata (RC), (RD) is provable, some consistent extensions of Γ contain the formula in question, other do not.

§ 5. Suppose $\ulcorner \mathcal{R}_\alpha \varphi \urcorner \in \Gamma$ or $\ulcorner \mathcal{R}_\alpha \psi \urcorner \in \Gamma$. In the former case, if the schema (RC) is provable, $\ulcorner \mathcal{R}_\alpha (\varphi \vee \psi) \urcorner \notin \Lambda$ and the condition (24) is satisfied. In the latter case, if the schema (RF) is provable, $\ulcorner \mathcal{R}_\alpha (\varphi \vee \psi) \urcorner \notin \Lambda$ and the condition (22) is satisfied. If neither of schemata (RC), (RD) is provable, some consistent extensions of Γ contain the formula inquestion, other do not.

§ 6. Suppose $\ulcorner \mathcal{R}_\alpha \varphi \urcorner, \ulcorner \mathcal{R}_\alpha \psi \urcorner \notin \Gamma$. If the schema (RH) is provable, $\ulcorner \mathcal{R}_\alpha (\varphi \vee \psi) \urcorner \notin \Lambda$ and the condition (26) is satisfied. Otherwise some consistent extensions of Γ contain the formula in question, other do not.

§ 7. Suppose $\ulcorner \mathcal{R}_\alpha \varphi \urcorner \notin \Gamma$ or $\ulcorner \mathcal{R}_\alpha \psi \urcorner \in \Gamma$. If either of the schemata (RI), (RJ) is provable, $\ulcorner \mathcal{R}_\alpha (\varphi \to \psi) \urcorner \in \Lambda$ and the condition (27) or (28), respectively, is satisfied. If neither of schemata (RI), (RJ) is provable, some consistent extensions of Γ contain the formula inquestion, other do not.

§ 8. Suppose $\ulcorner\mathcal{R}_\alpha\varphi\urcorner \in \Gamma$, $\ulcorner\mathcal{R}_\alpha\psi\urcorner \notin \Gamma$. If the schema (RK) is provable, $\ulcorner\mathcal{R}_\alpha(\varphi \rightarrow \rightarrow \psi)\urcorner \notin \Lambda$ and the condition (29) is satisfied. Otherwise some consistent extensions of Γ contain the formula in question, other do not.

§ 9. Suppose $\ulcorner\mathcal{R}_\alpha\varphi\urcorner$, $\ulcorner\mathcal{R}_\alpha\psi\urcorner \in \Gamma$. If the schema (RM) is provable, $\ulcorner\mathcal{R}_\alpha(\varphi \equiv \psi)\urcorner \in \Lambda$ and the condition (30) is satisfied. Otherwise some consistent extensions of Γ contain the formula in question, other do not.

§ 10. Suppose $\ulcorner\mathcal{R}_\alpha\varphi\urcorner$, $\ulcorner\mathcal{R}_\alpha\psi\urcorner \notin \Gamma$. If the schema (RN) is provable, $\ulcorner\mathcal{R}_\alpha(\varphi \equiv \psi)\urcorner \in \Lambda$ and the condition (31) is satisfied. Otherwise some consistent extensions of Γ contain the formula in question, other do not.

§ 11. Suppose $\ulcorner\mathcal{R}_\alpha\varphi\urcorner \in \Gamma$, $\ulcorner\mathcal{R}_\alpha\psi\urcorner \notin \Gamma$, or conversely. If either of schemata (RP), (RQ), respectively, is provable, $\ulcorner\mathcal{R}_\alpha(\varphi \equiv \psi)\urcorner \notin \Lambda$ and the conditions (32), (33), respectively, are satisfied. If neither of the schemata is provable, some consistent extensions of Γ contain the formula in question, other do not.

§ 12. Due to § 1–12, for every $\alpha \in \mathbb{PL}$, $\varphi \in \mathbb{QF}$, the formula $\ulcorner\mathcal{R}_\alpha\varphi\urcorner$ belongs to Λ if and only if $\mathfrak{d}(\alpha) \in \mathfrak{f}(\varphi)$. Hence, the claim is satisfied for all elementary formulas. Due to the presence of the axiom collection (1) and the derivative rule (MP) in every calculus from Zero to MR, as well as the truth-conditions (13), (14)–(18), the claim is also assured for compound formulas. The proof in this point is quite analogical to the proof of completeness of the classical propositional calculus, with atomic formulas playing the rôle of sentence letters. **QED**

The following crucial corollary to the theorems 4 and 5 states adequacy of all the calculi taken into consideration.

COROLLARY 6. *Every weak positional calculus, based on the axiom collection* (1) *and the derivation rule* (MP)*, as well as any arbitrary selection of axiom schemata* (RA)–(RQ)*, including the calculi* Zero *and* MR*, is adequate (i.e. both sound and complete).*

COROLLARY 7. *Every weak positional calculus, from* Zero *to* MR*, is decidable and the decision procedure is inherently provided by the construction of models.*

It is easily to check that there is exactly 32 768 different positional calculi, between MR and Zero, with respect to different distributive laws presented in this paper. The number varies of course depending on the set of connectives to be involved as well as relations to be assumed between the connectives. Let those calculi be called weak positional calculi.

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DISTRIBUTION LAWS IN WEAK POSITIONAL LOGICS

Summary

A formal language is positional if it involves a positional connecitve, i.e. a connective of realization to relate formulas to points of a kind, like points of realization or points of relativization. The connective in focus in this paper is the connective "$\mathcal{R}$", first introduced by Jerzy Łoś. Formulas $\ulcorner\mathcal{R}_\alpha\varphi\urcorner$ involve a singular name α and a formula φ to the effect that ϕ is satisfied (true) relative to the position designated by α. In weak positional calculi no nested occurences of the connective "$\mathcal{R}$" are allowed. The distribution problem in weak positional logics is actually the problem of distributivity of the connective "$\mathcal{R}$" over classical connectives, viz. the problem of relation between the occurences of classical connectives inside and outside the scope of the positional connective "$\mathcal{R}$".

PRAWA DYSTRYBUCYJNE W SŁABYCH LOGIKACH POZYCYJNYCH

Streszczenie

Logiki pozycyjne zawierają spójnik realizacji, który odnosi wyrażenie do pozycji ustalonego rodzaju, np. pozycji w czasie, przestrzeni, osób. W szczególności wyrażenie $\ulcorner\mathcal{R}_\alpha\varphi\urcorner$ należy odczytywać: w punkcie α jest tak, że φ lub w podobny odpowiedni sposób. Najsłabszą logiką pozycyjną, w której spójnik „$\mathcal{R}$" jest dystrybutywny względem wszystkich spójników klasycznego rachunku zdań, a konsekwentnie te spójniki są booleowskie w każdym kontekście, jest system MR. Rozważane w tej pracy słabe logiki pozycyjne są systemami pośrednimi między klasycznym rachunkiem zdań a systemem MR. Niektóre, ale niekoniecznie wszystkie, spójniki w tych systemach mogą być booleowskie. Przedstawiam tutaj prosty algorytm budowy dowolnego adekwatnego systemu z rozważanego przedziału, wyznaczonego przez wybrane prawa dystrybucyjne. Przedstawiony tutaj algorytm łatwo rozszerza się na inne zestawy spójników.

Key words: positional logic; weak positional logic; distribution; distributive law; realization connective; completeness.

Słowa kluczowe: logika pozycyjna; słaba logika pozycyjna; dystrybucja; prawo dystrybucyjne; realizacja connective; pełność.

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