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SET-THEORETIC SEMANTICS FOR MANY-VALUED POSITIONAL CALCULI

1. BACKGROUND

The connective of realization is used to relate expressions to contexts, be it possible worlds, moments in time, points in space, rational agents or else, generally referred to as positions. The first logic of the connective of realization (positional logic) was built by Jerzy Łoś for the needs of the methodology of natural sciences. The connective of realization was typically understood as a connective of relativized truth (or satisfaction). Such an understanding of realization is reflected in the distribution laws of the realization connective with respect to classic propositional connectives. Distribution laws mimic the usual truth conditions for compound expressions. It turns out, however, that notions of realization other than being true may require weaker assumptions (TKACZYK 2009). Łoś himself saw the possibility of using the tool he created to study other types of realizations, he also suggested a multi-valued interpretation of the connective of realization (1948), but he did not question the validity of the distribution laws. Nicolas Rescher (1971, 213–228) developed a positional temporal logic system with three values: truth, falsehood and gap (*indeterminacy*). Systematic research on the multi-valued interpretation of the logic of realization was undertaken by Marcin Tkaczyk (2013). He defined a general matrix model and also built and examined four positional calculi — $\mathsf{R}_{\mathfrak{B}}$, $\mathsf{R}_{\mathfrak{K}}$, $\mathsf{R}_{\mathfrak{P}}$, $\mathsf{R}_{\mathfrak{C}}$ — differing in terms of distribution of the realization connective with respect to negation. In a later

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work, Tkaczyk presented a simple algorithm for constructing adequate positional calculi with specific structures (TKACZYK 2018). Jarmużek (2007) considers other ways of weakening the assumptions regarding realization.

In the present paper I will show that the matrix systems $\mathsf{R}_{\mathfrak{B}}$, $\mathsf{R}_{\mathfrak{K}}$, $\mathsf{R}_{\mathfrak{P}}$ are not definable with the use of Tkaczyk's algorithm and I will formulate for them adequate Tkaczyk-style semantics (TKACZYK 2018). Section 2 introduces the weak positional language, sections 3 and 4 cover Tkaczyk's results concerning, respectively, set-theoretic and matrix semantics for positional calculi. Section 5 comprises the discussion of the relationship between the two approaches.

2. WEAK POSITIONAL LANGUAGE

The set of positional language symbols, in its most elementary variant, extends the alphabet of the classical propositional calculus by a symbol representing the realization operator and schematic names of positions. Therefore, this set consists of the following symbols:

– infinitely, but countably many schematic propositional letters: "p", "q", "r", "p_1", "p_2", ...;

– infinitely, but countably many schematic names (indicators), representing points of realization: "a", "b", "c", ...;

– propositional connectives: "$\neg$", "$\wedge$", "$\vee$", "$\rightarrow$", "$\equiv$", i.e. respectively the symbols of negation, conjunction, disjunction, implication, equivalence;

– positional operator, that is the connective of realization "$\mathcal{R}$";

– and brackets as punctuation marks.

In metalanguage we use the Greek letter "α" to denote individual names, "φ" and "ψ" to denote quasi-formulas, and "A" and "B" to denote arbitrary expressions of the constructed language. $\mathbb{IN}$ is the set of all individual names. We require that the set $\mathbb{IN}$ and the set of sentence letters are disjoint.

Defintion 1 (quasi-formula) *The set of quasi-formulas* $\mathbb{QF}$ *is the smallest set containing the set* $\mathbb{PL}$ *of schematic propositional letters and closed under the application of negation, conjunction, disjunction, implication and equivalence:*

- for any $\varphi \in \mathbb{PL}$, $\varphi \in \mathbb{QF}$;
- if $\varphi \in \mathbb{QF}$, then $(\neg\varphi) \in \mathbb{QF}$;
- if φ and $\psi \in \mathbb{QF}$, then $(\varphi \wedge \psi)$, $(\varphi \vee \psi)$, $(\varphi \rightarrow \psi)$, $(\varphi \equiv \psi) \in \mathbb{QF}$.

Therefore all schematic propositional letters, a negation of a quasi-formula, and a conjunction, disjunction, implication and equivalence of two quasi-formulas are quasi-formulas. $\mathbb{QF}$ is identical to the set of formulas PC. Quasi-formulas themselves are not formulas of a weak positional language, but they are included in atomic expressions as arguments of the realization connective.

Defintion 2 (atomic formula) *An atomic formula is any sign cluster*

$\mathcal{R}_\alpha\varphi,$

in which $\varphi \in \mathbb{QF}$, *and* $\alpha \in \mathbb{IN}$.

The quasi-formula φ in the formula $\mathcal{R}_\alpha\varphi$ is the scope of the connective "$\mathcal{R}$" in this formula, a ndividual name α is called an indicator and $\mathbb{AF}$ is the set of all atomic formulas. Compound formulas of the weak positional language are created with the use of propositional connectives of negation, conjunction, disjunction, implication and equivalence.

Defintion 3 (formula) *A set* $\mathbb{FOR}$ *of the formulas of the weak positional language is the smallest set such that:*

- *for any* $A \in \mathbb{AF}$, $A \in \mathbb{FOR}$;
- *for any* $A \in \mathbb{FOR}$, $(\neg A) \in \mathbb{FOR}$;
- *for any* $A, B \in \mathbb{FOR}$, $(A \wedge B)$, $(A \vee B)$, $(A \to B)$, $(A \equiv B) \in \mathbb{FOR}$.

We assume that the realiztion connective binds the most strongly among all constants present in the language, and for the remaining connectives, both in relation to quasi-formulass and formulas, the usual binding order applies, i.e. "$\neg$", "$\wedge$", "$\vee$", "$\to$", "$\equiv$". It is allowed to omit the outermost brackets and redundant inner brackets in formulas. In the case of complex quasi-formulas, the outer brackets are obligatory, but one can omit the redundant inner brackets, taking into account the binding order.

Note that in the $\mathbb{FOR}$ language, propositional connectives are syntactically ambiguous—they can combine symbols belonging to different categories (quasi-formulas or formulas) and form compound formulas if their arguments are formulas, or complex quasi-formulas if their arguments are quasi-formulas. We talk about *internal* usage of (or occurrence of) connectives when they occur within the scope of the "$\mathcal{R}$" operator, creating quasi-formulas, and about *external* usage (occurrence) when they appear outside this scope to form formulas. For example, in the formula

$\neg\mathcal{R}_\alpha(p \wedge q)$

the connective of negation is used externally and the connective of conjunction is used internally. The distribution laws of the realization operator with respect to propositional connectives:

$$\mathcal{R}_\alpha \neg\varphi \equiv \neg\mathcal{R}_\alpha\varphi, \tag{1}$$

$$\mathcal{R}_\alpha(\varphi \wedge \psi) \equiv \mathcal{R}_\alpha\varphi \wedge \mathcal{R}_\alpha\psi, \tag{2}$$

$$\mathcal{R}_\alpha(\varphi \vee \psi) \equiv \mathcal{R}_\alpha\varphi \vee \mathcal{R}_\alpha\psi, \tag{3}$$

$$\mathcal{R}_\alpha(\varphi \rightarrow \psi) \equiv \mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha\psi, \tag{4}$$

$$\mathcal{R}_\alpha(\varphi \equiv \psi) \equiv (\mathcal{R}_\alpha\varphi \equiv \mathcal{R}_\alpha\psi), \tag{5}$$

equate the external and internal use of these connectives. The minimal weak positional system in which all distribution laws are provable, i.e. the system **MR**, was described and examined by Tomasz Jarmużek and Andrzej Pietruszczak (2004) (JARMUŻEK and TKACZYK 2015 proposed alternative axiomatic and semantic approaches to the **MR** system).

3. PURELY DISTRIBUTIVE CALCULI

Let e be a substitution of the formulas $\mathbb{FOR}$ for propositional letters in formulas. The axiomatization of the systems considered by Tkaczyk comprises all axioms

$$e(\varphi), \textit{for any tautology } \varphi, \text{ of classical propositional calculus,} \tag{A0}$$

the rule of the schema *Modus Ponens*:

$$\frac{A \rightarrow B, A}{B.} \tag{MP}$$

and an arbitrary set of specific axioms among implications:

with respect to the connective of negation

$$\neg\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha\neg\varphi, \tag{RA}$$

$$\mathcal{R}_\alpha\neg\varphi \rightarrow \neg\mathcal{R}_\alpha\varphi, \tag{RB}$$

with respect to the connective of conjunction

$$\mathcal{R}_\alpha(\varphi \wedge \psi) \rightarrow \mathcal{R}_\alpha\varphi, \tag{RC}$$

$$\mathcal{R}_\alpha(\varphi \wedge \psi) \rightarrow \mathcal{R}_\alpha\psi, \tag{RD}$$

$$\mathcal{R}_\alpha\varphi \wedge \mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \wedge \psi), \tag{RE}$$

with respect to the connective of disjunction

$$\mathcal{R}_\alpha(\varphi \vee \psi) \rightarrow \mathcal{R}_\alpha\varphi \vee \mathcal{R}_\alpha\psi, \tag{RF}$$

$$\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha(\varphi \vee \psi), \tag{RG}$$

$$\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \vee \psi), \tag{RH}$$

with respect to the connective of implication

$$\neg\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha(\varphi \rightarrow \psi), \tag{RI}$$

$$\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \rightarrow \psi), \tag{RJ}$$

$$\mathcal{R}_\alpha(\varphi \rightarrow \psi) \rightarrow (\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha\psi), \tag{RK}$$

with respect to the connective of equivalence

$$\mathcal{R}_\alpha\varphi \wedge \mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \equiv \psi), \tag{RM}$$

$$\neg\mathcal{R}_\alpha\varphi \wedge \neg\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \equiv \psi), \tag{RN}$$

$$\mathcal{R}_\alpha(\varphi \equiv \psi) \rightarrow (\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha\psi), \tag{RP}$$

$$\mathcal{R}_\alpha(\varphi \equiv \psi) \rightarrow (\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha\varphi). \tag{RQ}$$

The formulas of the schemata (RA)–(RQ) we call implicational distribution laws. The schemata (RA) and (RB) taken together are deductively equivalent to distribution law (1), the schemata (RC), (RD) and (RE) are equivalent to (2), the schemata (RF), (RG) and (RH) are equivalent to (3), (RI), (RJ) and (RK) taken together are equivalent to (4), and (RM), (RN), (RP), (RQ) to (5) (Tkaczyk 2018, 170).

The systems so described—that is with the use of the schema (A0), selected distribution laws (RA)–(RQ) and from (MP) as the only primary rule—we shall call *purely distributional*. A system based only on axioms of the schema (A0) is the system Zero. The remaining systems are named after the distribution laws effective for them, for example BEHJN is a system of the axioms (A0), (RB), (RE), (RH), (RJ), (RN) and the rule (MP). The system ABCDEFGHIJLMNPQ can be shortened to A-Q. Tkaczyk proved that A-Q is deductively equivalent to the system **MR** (Tkaczyk 2018, 172).

Note that the lack of any assumptions about the meaning of connectives in the system Zero (their total non-extensionality) leads to quasi-formulas being completely unanalysable. Interpretation of atomic formulas is completely arbitrary and as a result the system is adequate with respect to the classical bivalent matrix.

Defintion 4 (set-theoretic model) *A set-theorical model of a weak positional language is a triple*

$$\mathfrak{M} = \langle \mathbb{U}, \mathfrak{d}, \mathfrak{f} \rangle, \tag{6}$$

such that

$$\mathbb{U} \neq \varnothing,$$
$$\mathfrak{d} : \mathbb{IN} \to \mathbb{U},$$
$$\mathfrak{f} : \mathbb{QF} \to \wp(\mathbb{U}).$$

$\mathbb{U}$ in the set-theoretic model is understood as a set of relativization points and $\mathfrak{d}(\alpha)$ is the unique designate of the individual name $\alpha \in \mathbb{IN}$. $\mathfrak{f}(\varphi)$ is a set of the relativization points from $\mathbb{U}$, in which the quasi-formula φ is satisfied. We stress that the function $\mathfrak{f}$ is totally arbitrary, that is its value for compound quasi-formulas does not have to be dependent on the value of the arguments of those quasi-formulas. For example, the interpretation of the quasi-formula $\varphi \in \mathbb{QF}$ and its negation $\neg\varphi$ can be in a certain model the same subset of the domain. Different classes of models in the set-theoretic semantics are determined by imposing additional conditions on the function $\mathfrak{f}$ (Tkaczyk 2018, 174–175).

The atomic formula $\mathcal{R}_\alpha\varphi$ is true in the set-theoretic model $\mathfrak{M}$ if and only if the designate of the individual name α belongs to the interpretation of the quasi-formula φ, symbolically

$$\mathfrak{M} \models \mathcal{R}_\alpha\varphi \textit{ iff } \mathfrak{d}(\alpha) \in \mathfrak{f}(\varphi), \tag{7}$$

for any $\alpha \in \mathbb{IN}$ and $\varphi \in \mathbb{QF}$. For compound formulas the (classical) conditions apply:

$$\mathfrak{M} \models \neg A \text{ iff } \mathfrak{M} \not\models A, \tag{8}$$

$$\mathfrak{M} \models A \wedge B \text{ iff } \mathfrak{M} \models A \text{ and } \mathfrak{M} \models B, \tag{9}$$

$$\mathfrak{M} \models A \vee B \text{ iff } \mathfrak{M} \models A \text{ or } \mathfrak{M} \models B, \tag{10}$$

$$\mathfrak{M} \models A \to B \text{ iff } \mathfrak{M} \not\models A \text{ or } \mathfrak{M} \models B, \tag{11}$$

$$\mathfrak{M} \models A \equiv B \text{ iff } \mathfrak{M} \models A, B \text{ or } \mathfrak{M} \not\models A, B. \tag{12}$$

Theorem 1 (Tkaczyk 2018) *Any purely distributional system is complete with respect to an appropriate set-theoretic semantics.*

The system `Zero` is complete with respect to the class of all models. Classes of models adequate to other purely distributional systems can be obtained algorithmically by imposing appropriate conditions on the $\mathfrak{f}$ function (TKACZYK 2018, 174–175):

with respect to the negation operator

$$\mathfrak{f}(\neg\varphi) \subseteq -\mathfrak{f}(\varphi), \tag{13}$$
$$-\mathfrak{f}(\varphi) \subseteq \mathfrak{f}(\neg\varphi), \tag{14}$$

with respect to the conjunction connective

$$\mathfrak{f}(\varphi \wedge \psi) \subseteq \mathfrak{f}(\varphi), \tag{15}$$
$$\mathfrak{f}(\varphi \wedge \psi) \subseteq \mathfrak{f}(\psi), \tag{16}$$
$$\mathfrak{f}(\varphi) \cap \mathfrak{f}(\psi) \subseteq \mathfrak{f}(\varphi \wedge \psi). \tag{17}$$

with respect to the connective of alternative

$$\mathfrak{f}(\varphi) \subseteq \mathfrak{f}(\varphi \vee \psi), \tag{18}$$
$$\mathfrak{f}(\psi) \subseteq \mathfrak{f}(\varphi \vee \psi), \tag{19}$$
$$\mathfrak{f}(\varphi \vee \psi) \subseteq \mathfrak{f}(\varphi) \cup \mathfrak{f}(\psi). \tag{20}$$

with respect to the connective of implication

$$-\mathfrak{f}(\varphi) \subseteq \mathfrak{f}(\varphi \rightarrow \psi), \tag{21}$$
$$\mathfrak{f}(\psi) \subseteq \mathfrak{f}(\varphi \rightarrow \psi), \tag{22}$$
$$\mathfrak{f}(\varphi \rightarrow \psi) \cap \mathfrak{f}(\varphi) \subseteq \mathfrak{f}(\psi), \tag{23}$$

with respect to the connective of equivalence

$$\mathfrak{f}(\varphi) \cap \mathfrak{f}(\psi) \subseteq \mathfrak{f}(\varphi \equiv \psi), \tag{24}$$
$$-\mathfrak{f}(\varphi) \cap -\mathfrak{f}(\psi) \subseteq \mathfrak{f}(\varphi \equiv \psi), \tag{25}$$
$$\mathfrak{f}(\varphi \equiv \psi) \cap \mathfrak{f}(\varphi) \subseteq \mathfrak{f}(\psi), \tag{26}$$
$$\mathfrak{f}(\varphi \equiv \psi) \cap \mathfrak{f}(\psi) \subseteq \mathfrak{f}(\varphi), \tag{27}$$

The conditions (13), (14) taken together characterize the classical connective of negation; (15), (16), (17)—conjunction; (18), (19), (20)—disjunction; (21), (22), (23)—inmplication; and (24), (25), (26), (27) classical connective of equivqlence (TKACZYK 2018, 174–175). Table 1 shows the relations between the classes of models determined by the properties of the satisfaction function and the truth of the distribution laws Observe that the truth condition (7) for the connective $\mathcal{R}$ is analogous to that of the hybrid satisfaction operator @ (cf. ARECES and TEN CATE). More detailed account of the relationship between hybrid and positional languages shall be given elsewhere.

Table 1: Distribution laws and the conditions concerning valuation

(RA) —(13),	(RF) —(18),	(RK) —(23),
(RB) —(14),	(RG) —(19),	(RM) —(24),
(RC) —(15),	(RH) —(20),	(RN) —(25),
(RD) —(16),	(RI) —(21),	(RP) —(26),
(RE) —(17),	(RJ) —(22),	(RQ) —(27).

4. MATRIX SYSTEMS

The general model for the weak positional language was defined in Tkaczyk (2013, 6–8).

Defintion 5 (matrix model) *A matrix model of the weak positional language is the quintuple*

$$\mathfrak{M} = \langle \mathbb{U}, \Omega, \Omega^*, \mathfrak{d}, \mathfrak{s} \rangle, \tag{28}$$

in which

$\mathbb{U} \neq \varnothing$,
$\Omega \neq \varnothing$,
$\Omega^* \subseteq \Omega$,
$\mathfrak{d} : \mathbb{IN} \rightarrow \mathbb{U}$,
$\mathfrak{s} : \mathbb{U} \times \mathbb{QF} \rightarrow \Omega$

where the values of function $\mathfrak{s}$ for $\langle u, \varphi \rangle$, $u \in \mathbb{U}$ and a quasi-formula φ is determined by operations—unary $\mathfrak{f}^{\neg}$ and binary $\mathfrak{f}^{\wedge}, \mathfrak{f}^{\vee}, \mathfrak{f}^{\rightarrow}, \mathfrak{f}^{\equiv}$—in the set Ω in the following way:

$$\mathfrak{s}(x, \neg\varphi) = \mathfrak{f}^{\neg}(\mathfrak{s}(x, \varphi)), \tag{29}$$

$$\mathfrak{s}(x, \varphi \wedge \psi) = \mathfrak{f}^{\wedge}(\mathfrak{s}(x, \varphi), \mathfrak{s}(x, \psi)), \tag{30}$$

$$\mathfrak{s}(x, \varphi \vee \psi) = \mathfrak{f}^{\vee}(\mathfrak{s}(x, \varphi), \mathfrak{s}(x, \psi)), \tag{31}$$

$$\mathfrak{s}(x, \varphi \rightarrow \psi) = \mathfrak{f}^{\rightarrow}(\mathfrak{s}(x, \varphi), \mathfrak{s}(x, \psi)), \tag{32}$$

$$\mathfrak{s}(x, \varphi \equiv \psi) = \mathfrak{f}^{\equiv}(\mathfrak{s}(x, \varphi), \mathfrak{s}(x, \psi)). \tag{33}$$

$\mathbb{U}$ i $\mathfrak{d}$ are the same as in the set-theoretic model, Ω is a set of logical values and its subset Ω^* a set of designated values. An atomic formula $\mathcal{R}_\alpha\varphi$ is true in a matrix model $\mathfrak{M}$ if and only if the function $\mathfrak{s}$ of the model $\mathfrak{M}$ assigns a designated value to the quasi-formula φ in the point determined by α.

$$\mathfrak{M} \vDash \mathcal{R}_{\alpha}\varphi \textit{ iff } \mathfrak{s}(\mathfrak{d}(\alpha),\varphi) \in \Omega^{*}. \tag{34}$$

The truth conditions of the compound formulas are classial—(8)–(12).

Tkaczyk constructed and examined four systems in $\mathbb{FOR}$—$\mathsf{R}_{\mathfrak{B}}$, $\mathsf{R}_{\mathfrak{K}}$, $\mathsf{R}_{\mathfrak{P}}$, $\mathsf{R}_{\mathfrak{C}}$—different differing in terms of the distribution of "$\mathcal{R}$" over the connective of negation. The system $\mathsf{R}_{\mathfrak{B}}$ is based on the axioms (A0), (2),

$$\mathcal{R}_{\alpha}\neg(\varphi \wedge \psi) \equiv \mathcal{R}_{\alpha}\neg\varphi \vee \mathcal{R}_{\alpha}\neg\psi, \tag{35}$$

and the primitive rules: (MP) and rules of mutual interchange of quasi-formulas of the schemata:

$$\neg\neg\varphi \,\|\, \varphi, \tag{36}$$

$$\varphi \vee \psi \,\|\, \neg(\neg\varphi \wedge \neg\psi), \tag{37}$$

$$\varphi \to \psi \,\|\, \neg(\varphi \wedge \neg\psi), \tag{38}$$

$$\varphi \equiv \psi \,\|\, (\varphi \to \psi) \wedge (\psi \to \varphi), \tag{39}$$

for any $\varphi,\psi \in \mathbb{QF}$. "$\|$" is the symbol of mutual interchange. The system $\mathsf{R}_{\mathfrak{B}}$ is adequate with respect to the class of models $\mathfrak{B}$, in which:

$$\Omega_{\mathfrak{B}} = \{1,0,X,Y\}, \Omega^{*}_{\mathfrak{B}} = \{1,X\}, \tag{40}$$

and the operations $\mathfrak{f}^{\neg}_{\mathfrak{B}}$, $\mathfrak{f}^{\wedge}_{\mathfrak{B}}$, $\mathfrak{f}^{\vee}_{\mathfrak{B}}$, $\mathfrak{f}^{\to}_{\mathfrak{B}}$, $\mathfrak{f}^{\equiv}_{\mathfrak{B}}$ are determined as in table 2.

Table 2: Operations $\mathfrak{f}^{\neg}_{\mathfrak{B}}$, $\mathfrak{f}^{\wedge}_{\mathfrak{B}}$, $\mathfrak{f}^{\vee}_{\mathfrak{B}}$, $\mathfrak{f}^{\to}_{\mathfrak{B}}$, $\mathfrak{f}^{\equiv}_{\mathfrak{B}}$

	¬
1	0
X	X
Y	Y
0	1

∧	1	X	Y	0
1	1	X	Y	0
X	X	X	0	0
Y	Y	0	Y	0
0	0	0	0	0

∨	1	X	Y	0
1	1	1	1	1
X	1	X	1	X
Y	1	1	Y	Y
0	1	X	Y	0

→	1	X	Y	0
1	1	X	Y	0
X	1	X	1	X
Y	1	1	Y	Y
0	1	1	1	1

≡	1	X	Y	0
1	1	X	Y	0
X	X	X	1	X
Y	Y	1	Y	Y
0	0	X	Y	1

It is easy to check that no distribution law is a tautology of the system $\mathsf{R}_{\mathfrak{B}}$. Let us consider the formula

$$\mathcal{R}_{a}\neg p \to \neg\mathcal{R}_{a}p, \tag{41}$$

of the schema (RB). Let $\mathfrak{s}(\mathfrak{d}(a), p) = X$ in a certain model $\mathfrak{M} \in \mathfrak{B}$. Then both the formula "$\mathcal{R}_{a}p$" and the formula "$\mathcal{R}_{a}\neg p$" (since $\mathfrak{f}^{\neg}_{\mathfrak{B}}(X) = X$) are true in $\mathfrak{M}$. But since "$\mathcal{R}_{a}p$" is true then its negation "$\neg\mathcal{R}_{a}p$" will be false in the model $\mathfrak{M}$. Thus the interpretation is a countermodel for (RB). Let us now take into consideration the formula of the form (RA):

$$\neg\mathcal{R}_a p \rightarrow \mathcal{R}_a \neg p, \tag{42}$$

If in the model $\mathfrak{M} \in \mathfrak{B}$, $\mathfrak{s}(\mathfrak{d}(a), p) = Y$, then to "$\mathcal{R}_a p$" will be false in $\mathfrak{M}$, while the formula "$\neg\mathcal{R}_a p$", that is the antecedent of the implication (42), will be true. However, since $\mathfrak{f}_{\mathfrak{B}}^{\neg}(Y) = Y$, then the formula to "$\mathcal{R}_a \neg p$" will be false in the model $\mathfrak{M}$, so the implication will also be false (42). The system $R_{\mathfrak{K}}$ can be obtained by the extension of the system $R_{\mathfrak{B}}$ by schema (RB). The set $\mathfrak{K}$ of models, adequate with respect to $R_{\mathfrak{K}}$, is determined by the sets of values:

$$\Omega_{\mathfrak{K}} = \{1, 0, Y\}, \Omega_{\mathfrak{K}}^{*} = \{1\}, \tag{43}$$

and the set of operations $\mathfrak{f}_{\mathfrak{K}}^{\neg}, \mathfrak{f}_{\mathfrak{K}}^{\wedge}, \mathfrak{f}_{\mathfrak{K}}^{\vee}, \mathfrak{f}_{\mathfrak{K}}^{\rightarrow}, \mathfrak{f}_{\mathfrak{K}}^{\equiv}$, described in table 3.

Table 3: Operations $\mathfrak{f}_{\mathfrak{K}}^{\neg}, \mathfrak{f}_{\mathfrak{K}}^{\wedge}, \mathfrak{f}_{\mathfrak{K}}^{\vee}, \mathfrak{f}_{\mathfrak{K}}^{\rightarrow}, \mathfrak{f}_{\mathfrak{K}}^{\equiv}$

	¬
1	0
Y	Y
0	1

∧	1	Y	0
1	1	Y	0
Y	Y	Y	0
0	0	0	0

∨	1	Y	0
1	1	1	1
Y	1	Y	Y
0	1	Y	0

→	1	Y	0
1	1	Y	0
Y	1	Y	Y
0	1	1	1

≡	1	Y	0
1	1	Y	0
Y	Y	Y	Y
0	0	Y	1

Obviously in the system $\mathsf{R}_{\mathfrak{K}}$ the distribution law (RA) does not apply. The countermodel is the same as in the case of $\mathsf{R}_{\mathfrak{B}}$.

The system $\mathsf{R}_{\mathfrak{P}}$ is obtained through adding to $\mathsf{R}_{\mathfrak{B}}$ schema (RA). $R_{\mathfrak{P}}$ is adequate with respect to the class of models $\mathfrak{P}$:

$$\Omega_{\mathfrak{P}} = \{1, 0, X\}, \Omega_{\mathfrak{P}}^{*} = \{1, X\}, \tag{44}$$

the operations $\mathfrak{f}_{\mathfrak{P}}^{\neg}, \mathfrak{f}_{\mathfrak{P}}^{\wedge}, \mathfrak{f}_{\mathfrak{P}}^{\vee}, \mathfrak{f}_{\mathfrak{P}}^{\rightarrow}, \mathfrak{f}_{\mathfrak{P}}^{\equiv}$ in the set $\Omega_{\mathfrak{P}}$ are characterized in table 4.

Table 4: Opertions $\mathfrak{f}_{\mathfrak{P}}^{\neg}, \mathfrak{f}_{\mathfrak{P}}^{\wedge}, \mathfrak{f}_{\mathfrak{P}}^{\vee}, \mathfrak{f}_{\mathfrak{P}}^{\rightarrow}, \mathfrak{f}_{\mathfrak{P}}^{\equiv}$

	¬
1	0
X	X
0	1

∧	1	X	0
1	1	X	0
X	X	X	0
0	0	0	0

∨	1	X	0
1	1	1	1
X	1	X	X
0	1	X	0

→	1	X	0
1	1	X	0
X	1	X	X
0	1	1	1

≡	1	X	0
1	1	X	0
X	X	X	X
0	0	X	1

Let us note, that formally the class of models $\mathfrak{P}$ differs from the class $\mathfrak{K}$ only in the set of designated values—in the former the designated value is, besides truth, the non-classical value X. Therefore the distribution law (RA) is not a chema of a tautology of the class of models P. A countermodel for the formula of schema (RA) is, as in $\mathsf{R}_{\mathfrak{B}}$, such an interpretation, in which for a certain $\varphi \in \mathbb{QF}$, $\alpha \in \mathbb{IN}$, $\mathfrak{s}(\mathfrak{d}(\alpha), \varphi) = X$.

In the system $\mathsf{R}_{\mathfrak{C}}$ the axiom schemata ($\mathtt{A0}$), (2) and all the rules of the system $\mathsf{R}_{\mathfrak{B}}$ and, additionally, the distribution law (1) are accepted. $\mathsf{R}_{\mathfrak{C}}$ is adequate with respect to the class of models $\mathfrak{C}$, determined by the sets of values:

$$\Omega_{\mathfrak{C}} = \{1,0\}, \Omega^{*}_{\mathfrak{C}} = \{1\}, \tag{45}$$

with the operations $\mathfrak{f}^{\neg}_{\mathfrak{C}}$, $\mathfrak{f}^{\wedge}_{\mathfrak{C}}$, $\mathfrak{f}^{\vee}_{\mathfrak{C}}$, $\mathfrak{f}^{\rightarrow}_{\mathfrak{C}}$, $\mathfrak{f}^{\equiv}_{\mathfrak{C}}$ in the set $\Omega_{\mathfrak{C}}$, presented in table 5:

Table 5: Operations $\mathfrak{f}^{\neg}_{\mathfrak{C}}$, $\mathfrak{f}^{\wedge}_{\mathfrak{C}}$, $\mathfrak{f}^{\vee}_{\mathfrak{C}}$, $\mathfrak{f}^{\rightarrow}_{\mathfrak{C}}$, $\mathfrak{f}^{\equiv}_{\mathfrak{C}}$

	$\neg$
1	0
0	1

$\wedge$	1	0
1	1	0
0	0	0

$\vee$	1	0
1	1	1
0	1	0

$\rightarrow$	1	0
1	1	0
0	1	1

$\equiv$	1	0
1	1	0
0	0	1

Theorem 2 (completeness) *Each of the systems* $\mathsf{R}_{\mathfrak{B}}$, $\mathsf{R}_{\mathfrak{K}}$, $\mathsf{R}_{\mathfrak{P}}$, $\mathsf{R}_{\mathfrak{C}}$ *is complete with respect to its matrix semantics* (TKACZYK 2013).

$\mathsf{R}_{\mathfrak{C}}$ is deductively equivalent to the system **MR** (TKACZYK 2013, 18), so the realization operator is there completely distributive over all propositional connectives. For the sake of further considerations we shall introduce distribution laws in the systems $\mathsf{R}_{\mathfrak{B}}$, $\mathsf{R}_{\mathfrak{K}}$ and $\mathsf{R}_{\mathfrak{P}}$. Because the systems $\mathsf{R}_{\mathfrak{K}}$ and $\mathsf{R}_{\mathfrak{P}}$ are extensions of the system $\mathsf{R}_{\mathfrak{B}}$, every theorem of $\mathsf{R}_{\mathfrak{B}}$ is also a theorem of each $\mathsf{R}_{\mathfrak{K}}$, $\mathsf{R}_{\mathfrak{P}}$.

$$\vdash_{R_{\mathfrak{B}}} \mathcal{R}_{\alpha}\varphi \rightarrow \mathcal{R}_{\alpha}(\varphi \vee \psi) \tag{46}$$

Proof:

1. $\mathcal{R}_{\alpha}\neg(\neg\varphi \wedge \neg\psi) \equiv \mathcal{R}_{\alpha}(\neg\neg\varphi) \vee \mathcal{R}_{\alpha}(\neg\neg\psi)$ (35)
2. $\mathcal{R}_{\alpha}(\neg\neg\varphi) \vee \mathcal{R}_{\alpha}(\neg\neg\psi) \rightarrow \mathcal{R}_{\alpha}\neg(\neg\varphi \wedge \neg\psi)$ 1, (A0)
3. $\mathcal{R}_{\alpha}\varphi \vee \mathcal{R}_{\alpha}\psi \rightarrow \mathcal{R}_{\alpha}(\varphi \vee \psi)$ 2, (36), (37)
4. $(\mathcal{R}_{\alpha}\varphi \vee \mathcal{R}_{\alpha}\psi \rightarrow \mathcal{R}_{\alpha}(\varphi \vee \psi)) \rightarrow (\mathcal{R}_{\alpha}\varphi \rightarrow \mathcal{R}_{\alpha}(\varphi \vee \psi))$ (A0)
5. $\mathcal{R}_{\alpha}\varphi \rightarrow \mathcal{R}_{\alpha}(\varphi \vee \psi)$ 4, 3× (MP)

Analogously we prove the implication (RH).

$$\vdash_{R_{\mathfrak{B}}} \mathcal{R}_{\alpha}(\varphi \vee \psi) \rightarrow \mathcal{R}_{\alpha}\varphi \vee \mathcal{R}_{\alpha}\psi \tag{47}$$

Proof:

1.	$\mathcal{R}_\alpha\neg(\neg\varphi \wedge \neg\psi) \equiv \mathcal{R}_\alpha(\neg\neg\varphi) \vee \mathcal{R}_\alpha(\neg\neg\psi)$	(35)
2.	$\mathcal{R}_\alpha\neg(\neg\varphi \wedge \neg\psi) \rightarrow \mathcal{R}_\alpha(\neg\neg\varphi) \vee \mathcal{R}_\alpha(\neg\neg\psi)$	1, (A0)
3.	$\mathcal{R}_\alpha(\varphi \vee \psi) \rightarrow \mathcal{R}_\alpha(\varphi) \vee \mathcal{R}_\alpha\psi)$	2, (37), (36)

$$\vdash_{R_{\mathfrak{B}}} \mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \rightarrow \psi) \tag{48}$$

Proof:

1.	$\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\neg\varphi \vee \psi)$	(RH)
2.	$\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha\neg(\neg\neg\varphi \wedge \neg\psi)$	1, (37)
3.	$\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha\neg(\varphi \wedge \neg\psi)$	2, (36)
4.	$\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \rightarrow \psi)$	3, (38)

$$\vdash_{R_{\mathfrak{B}}} \mathcal{R}_\alpha\varphi \wedge \mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \equiv \psi) \tag{49}$$

Proof:

1.	$\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \rightarrow \psi)$	thesis (RJ)
2.	$\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha(\psi \rightarrow \varphi)$	thesis (RJ)
3.	$\mathcal{R}_\alpha\varphi \wedge \mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\psi \rightarrow \varphi) \wedge \mathcal{R}_\alpha(\varphi \rightarrow \psi)$	1, 2, (A0)
4.	$\mathcal{R}_\alpha(\psi \rightarrow \varphi) \wedge \mathcal{R}_\alpha(\varphi \rightarrow \psi) \equiv \mathcal{R}_\alpha((\psi \rightarrow \varphi) \wedge (\varphi \rightarrow \psi))$	(2)
5.	$\mathcal{R}_\alpha\varphi \wedge \mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha((\psi \rightarrow \varphi) \wedge (\varphi \rightarrow \psi))$	3, 4
6.	$\mathcal{R}_\alpha\varphi \wedge \mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \equiv \psi)$	5, (39)

$$\vdash_{R_{\mathfrak{K}}} \mathcal{R}_\alpha(\varphi \rightarrow \psi) \rightarrow (\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha\psi), \tag{50}$$

Proof:

1.	$\mathcal{R}_\alpha\neg(\varphi \wedge \neg\psi) \equiv \mathcal{R}_\alpha\neg\varphi \vee \mathcal{R}_\alpha\neg\neg\psi$	(35)
2.	$\mathcal{R}_\alpha\neg(\varphi \wedge \neg\psi) \rightarrow (\mathcal{R}_\alpha\neg\varphi \vee \mathcal{R}_\alpha\neg\neg\psi)$	1, (A0)
3.	$\mathcal{R}_\alpha(\varphi \rightarrow \psi) \rightarrow \mathcal{R}_\alpha\neg\varphi \vee \mathcal{R}_\alpha\psi$	2, (38),(36)
4.	$\mathcal{R}_\alpha\neg\varphi \rightarrow \neg\mathcal{R}_\alpha\varphi$	(RB)
5.	$\mathcal{R}_\alpha(\varphi \rightarrow \psi) \rightarrow \neg\mathcal{R}_\alpha\varphi \vee \mathcal{R}_\alpha\psi$	3,4, (A0)
6.	$\mathcal{R}_\alpha(\varphi \rightarrow \psi) \rightarrow (\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha\psi)$	5, (A0)

$$\vdash_{R_{\mathfrak{K}}} \mathcal{R}_\alpha(\varphi \equiv \psi) \rightarrow (\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha\psi) \tag{51}$$

Proof:

1. $\mathcal{R}_\alpha(\varphi \rightarrow \psi) \rightarrow (\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha\psi)$ (RK)
2. $\mathcal{R}_\alpha(\varphi \rightarrow \psi) \wedge \mathcal{R}_\alpha(\psi \rightarrow \varphi) \rightarrow (\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha\psi)$ 1, (A0)
3. $\mathcal{R}_\alpha((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)) \rightarrow (\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha\psi)$ 2, (2)
4. $\mathcal{R}_\alpha(\varphi \equiv \psi) \rightarrow (\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha\psi)$ 3,(39)

Analogously we prove in $R_{\mathfrak{K}}$ implication (RQ).

$$\vdash_{R_{\mathfrak{P}}} \neg\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha(\varphi \rightarrow \psi) \tag{52}$$

Proof:

1. $\mathcal{R}_\alpha(\varphi \wedge \neg\psi) \rightarrow \mathcal{R}_\alpha\varphi$ (RG)
2. $\neg\mathcal{R}_\alpha\varphi \rightarrow \neg\mathcal{R}_\alpha(\varphi \wedge \neg\psi)$ 1, (A0)
3. $\neg\mathcal{R}_\alpha(\varphi \wedge \neg\psi) \rightarrow \mathcal{R}_\alpha\neg(\varphi \wedge \neg\psi)$ (RA)
4. $\neg\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha\neg(\varphi \wedge \neg\psi)$ 2, 3, (A0)
5. $\neg\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha(\varphi \rightarrow \psi)$ 4, (38)

$$\vdash_{R_{\mathfrak{P}}} \neg\mathcal{R}_\alpha\varphi \wedge \neg\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \equiv \psi) \tag{53}$$

Proof:

1. $\neg\mathcal{R}_\alpha\varphi \rightarrow \mathcal{R}_\alpha(\varphi \rightarrow \psi)$ (RI)
2. $\neg\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\psi \rightarrow \varphi)$ (RI)
3. $\neg\mathcal{R}_\alpha\varphi \wedge \neg\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \rightarrow \psi) \wedge \mathcal{R}_\alpha(\psi \rightarrow \varphi)$ 1, 2, (A0)
4. $\mathcal{R}_\alpha((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)) \equiv \mathcal{R}_\alpha(\varphi \rightarrow \psi) \wedge \mathcal{R}_\alpha(\psi \rightarrow \varphi)$ (2)
5. $\neg\mathcal{R}_\alpha\varphi \wedge \neg\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha((\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi))$ 3, 4, (A0)
6. $\neg\mathcal{R}_\alpha\varphi \wedge \neg\mathcal{R}_\alpha\psi \rightarrow \mathcal{R}_\alpha(\varphi \equiv \psi)$ 5, (39)

5. SET-THEORETIC SEMANTICS FOR THE SYSTEMS $\mathsf{R}_{\mathfrak{B}}$, $\mathsf{R}_{\mathfrak{K}}$, $\mathsf{R}_{\mathfrak{P}}$, $\mathsf{R}_{\mathfrak{C}}$

Because the system $\mathsf{R}_{\mathfrak{K}}$ is equivalent to the system **MR** (section 4) its adequate semantics is the class of models fulfilling all conditions (13)–(27). On the other hand the systems $\mathsf{R}_{\mathfrak{B}}$, $\mathsf{R}_{\mathfrak{K}}$, $\mathsf{R}_{\mathfrak{P}}$ cannot be characterized with the use of a combination of the conditions (13)–(27) from section 3. It means that they are not equivalent to any purely distributional calculus.

Theorem 3 *No class of set-theoretic models determined by a certain set of conditions from among* (13)—(27) *defines* $\mathsf{R}_{\mathfrak{B}}$*,* $\mathsf{R}_{\mathfrak{K}}$ *nor* $\mathsf{R}_{\mathfrak{P}}$.

Proof If any class of set-theoretic models $\mathfrak{k}$ (meeting the specified conditions) defined any of the systems $\mathsf{R}_{\mathfrak{B}}$, $\mathsf{R}_{\mathfrak{K}}$, $\mathsf{R}_{\mathfrak{P}}$, then the tautologies of the class $\mathfrak{k}$ would have to be all and only the theorems of that system. Let us consider then the schema (35), which is a theorem of all those systems. The schema (35) is tautological in a class of models $\mathfrak{k}$ if and only if $\mathfrak{k}$ jointly satisfies the conditions (13), (14), (15), (16) and (17). We shall see that it really is so.

1. If the conditions (13), (14), (15), (16) and (17) are satisfied, then the connective of negation and conjunction correspond to operations $-$ and $\cap$. Then $\mathfrak{d}(\alpha) \in \mathfrak{f}(\neg(\varphi \wedge \psi))$ iff $\mathfrak{d}(\alpha) \notin \mathfrak{f}((\varphi \wedge \psi))$ iff $\mathfrak{d}(\alpha) \notin \mathfrak{f}(\varphi)$ or $\mathfrak{d}(\alpha) \notin \mathfrak{f}(\psi)$ iff $\mathfrak{d}(\alpha) \in \mathfrak{f}(\neg\varphi)$ or $\mathfrak{d}(\alpha) \in \mathfrak{f}(\neg\psi)$ and any formula of schema (35) is true.

2. If at least one of the conditions is not met, then (35) is not tautological. Since the connective $\mathcal{R}$ is fully distributive over the connective of conjunction in all considered systems (that is due to the axiom (2)), only the conditions regarding negation will be dealt with. We shall assume throughout that all conditions (15), (16), (17) are met and omit the discussion of the cases of them being not satisfied.

– Consider first the condition (13). If (13) holds, then $\mathfrak{f}(\varphi) \cap \mathfrak{f}(\neg\varphi) = \varnothing$, for every quasi-formula φ. Let us assume that (13) does not hold. Thus there exists a model such that for some $u \in \mathbb{U}$, both $u \in \mathfrak{f}(\neg(\varphi \wedge \psi))$ and $u \in \mathfrak{f}(\varphi \wedge \psi)$. By (15) and (16) we have $u \in \mathfrak{f}(\varphi)$ and $u \in \mathfrak{f}(\psi)$. We may stipulate further that $u \notin \mathfrak{f}(\neg\varphi)$ and $u \in \mathfrak{f}(\neg\psi)$, and this constitutes a counter-model to (35).

– If, on the other hand, (14) does not hold, there exists a model such that for some quasi-formula φ it is not the case that $\mathfrak{f}(\varphi) \cap \mathfrak{f}(\neg\varphi) = \mathbb{U}$. Put then

$u \notin \mathfrak{f}(\varphi \wedge \psi)$ and $u \notin \mathfrak{f}(\neg(\varphi \wedge \psi))$. By (15) and (16) we get $u \notin \mathfrak{f}(\varphi)$ or $u \notin \mathfrak{f}(\psi)$. Let moreover $u \in \mathfrak{f}(\neg\varphi)$ (or $u \in \mathfrak{f}(\neg\psi)$, which gives us a countermodel to (35).

However, if both conditions (13), (14) required for the truth of the formulas (35) apply, then any formula of schema (1) is a tautology. Since in all systems $\mathsf{R}_{\mathfrak{B}}$, $\mathsf{R}_{\mathfrak{K}}$, $\mathsf{R}_{\mathfrak{P}}$ there exists a countermodel for (1) (see paragraph 4) and in consequence the schema cannot be a schema of the thesis it entails that none of the systems $\mathsf{R}_{\mathfrak{B}}$, $\mathsf{R}_{\mathfrak{K}}$, $\mathsf{R}_{\mathfrak{P}}$ can be both sound and complete with respect to set-theoretic semantics determined by the combination of the conditions (13)–(27). **QED**

Thus, in order to build an adequate set-theoretic semantics for the systems, additional conditions concerning the meaning of the propositional connectives should be used. Let us note that any of the systems $\mathsf{R}_{\mathfrak{B}}$, $\mathsf{R}_{\mathfrak{K}}$, $\mathsf{R}_{\mathfrak{P}}$, $\mathsf{R}_{\mathfrak{C}}$ extends a certain purely distributive system by the rules (36)–(39).

Lemma 4 *The system System* $\mathsf{R}_{\mathfrak{B}}$ *is equivalent to the system* `CDEFGH` *with the mutual interchange rules* (36)–(39).

Proof Implication schemata (`RC`), (`RD`), (`RE`) can easily be derived from axiom (2), proofs of implication (`RF`), (`RG`), (`RH`) in the system $R_{\mathfrak{B}}$ were given in the previous section. On the other hand distributive laws (2), (3) are the theses of `CDEFGH` with mutual interchange rules (36)–(39) (respectively from (`RC`), (`RD`), (`RE`) or (`RF`), (`RG`), (`RH`) and (`A0`)). From the last one the schema (35) is derived in the following way:

1. $\mathcal{R}_\alpha(\neg\varphi \vee \neg\psi) \equiv \mathcal{R}_\alpha\neg\varphi \vee \mathcal{R}_\alpha\neg\psi$ (3)
2. $\mathcal{R}_\alpha\neg(\neg\neg\varphi \wedge \neg\neg\psi) \equiv \mathcal{R}_\alpha\neg\varphi \vee \mathcal{R}_\alpha\neg\psi$ 1, (37)
3. $\mathcal{R}_\alpha\neg(\varphi \wedge \psi) \equiv \mathcal{R}_\alpha\neg\varphi \vee \mathcal{R}_\alpha\neg\psi$ 2, (36)

QED

Lemma 5 *The system System* $\mathsf{R}_{\mathfrak{K}}$ *is equivalent to the system ACDEFGH with mutual interchange rules* (36)–(39).

Proof The lemma follows from lemma 4. It is sufficient to notice that the system $\mathsf{R}_{\mathfrak{K}}$ is an extension of the system $\mathsf{R}_{\mathfrak{B}}$ by schema (`RA`). **QED**

Lemma 6 *The system System* $\mathsf{R}_{\mathfrak{P}}$ *is equivalent to the system BCDEFGH with mutual interchange rules* (36)–(39).

Proof The lemma follows from lemma 4 for the system $R_{\mathfrak{P}}$ is an extension of the system $\mathsf{R}_{\mathfrak{B}}$ by the schema (RB). **QED**

Additionally we can formulate an analogous lemma concerning the system $\mathsf{R}_{\mathfrak{C}}$.

Lemma 7 *The system System* $\mathsf{R}_{\mathfrak{C}}$ *is equivalent to the system* ABCDEFGH *with mutual interchange rules* (36)–(39).

It can now be seen that besides the appropriate conditions concerning the distributive laws (as presented in section 3), it is sufficient to accept the conditions corresponding to the mutual interchange rules:

$$\mathfrak{f}(\varphi) = \mathfrak{f}(\varphi^*), \quad (54a)$$

where

$$\varphi^* = \varphi(\neg\neg\varphi \parallel \varphi) \quad (54b)$$

$$\text{or } \varphi^* = \varphi(\varphi \vee \psi \parallel \neg(\neg\varphi \wedge \neg\psi)) \quad (54c)$$

$$\text{or } \varphi^* = \varphi(\varphi \rightarrow \psi \parallel \neg(\varphi \wedge \neg\psi)) \quad (54d)$$

$$\text{or } \varphi^* = \varphi(\varphi \equiv \psi \parallel (\varphi \rightarrow \psi) \wedge (\psi \rightarrow \varphi)). \quad (54e)$$

The set-theoretic semantics for the systems will thus be defined by the following conditions:

$-\mathsf{R}_{\mathfrak{B}}$: (15), (16), (17), (18), (19), (20), (5),

$-\mathsf{R}_{\mathfrak{K}}$: (13), (15), (16), (17), (18), (19), (20), (5),

$-\mathsf{R}_{\mathfrak{P}}$: (14), (15), (16), (17), (18), (19), (20), (5),

$-\mathsf{R}_{\mathfrak{C}}$: (13), (14), (15), (16), (17), (18), (19), (20), (5),

Theorem 8 (soundness) *The systems* $\mathsf{R}_{\mathfrak{B}}, \mathsf{R}_{\mathfrak{K}}, \mathsf{R}_{\mathfrak{P}}, \mathsf{R}_{\mathfrak{C}}$ *are sound with respect to their set-theoretic semantics.*

Proof Regarding the system $\mathsf{R}_{\mathfrak{B}}$, let us note that the axioms (A0) are true in any model, and the rule (MP) inherits soundness, through the classical definitions of the truthfulness of compound formulas; conditions (15), (16), (17) jointly ensure that the axioms are tautologies (2). According to (5), quasi-formulas which are mutually replaceable under rules (36)–(39) have the same interpretation in the model, which, together with definition (7), guarantees, that the mutual interchange rules preserve truth. Also axioms (35) are tautologies of the class of models $\mathsf{R}_{\mathfrak{B}}$. If we assume, that

$\mathfrak{M} \vDash \mathcal{R}_\alpha\neg(\varphi \wedge \psi)$, then $\mathfrak{d}(\alpha) \in \mathfrak{f}(\neg(\varphi \wedge \psi))$. From (54) $\mathfrak{d}(\alpha) \in \mathfrak{f}(\neg(\neg\neg\varphi \wedge \neg\neg\psi))$. From (55) $\mathfrak{d}(\alpha) \in \mathfrak{f}(\neg\varphi \vee \neg\psi)$. From (18), (19), (20) $\mathfrak{d}(\alpha) \in \mathfrak{f}(\neg\varphi) \cup \mathfrak{f}(\neg\psi)$. Thus we have that $\mathfrak{M} \vDash \mathcal{R}_\alpha\neg\varphi$ or $\mathfrak{M} \vDash \mathcal{R}_\alpha\neg\psi$, so $\mathfrak{M} \vDash \mathcal{R}_\alpha\neg\varphi \vee \mathcal{R}_\alpha\neg\psi$. On the other hand under condition that $\mathfrak{M} \nvDash \mathcal{R}_\alpha\neg(\varphi \wedge \psi)$, $\mathfrak{d}(\alpha) \notin \mathfrak{f}(\neg(\varphi \wedge \psi))$ and, on the basis of transformations analogous the previous ones, we get $\mathfrak{d}(\alpha) \notin \mathfrak{f}(\neg\varphi) \cup \mathfrak{f}(\neg\psi)$. Thus $\mathfrak{d}(\alpha) \notin \mathfrak{f}(\neg\varphi)$ and $\mathfrak{d}(\alpha) \notin \mathfrak{f}(\neg\psi)$. Thus we have $\mathfrak{M} \nvDash \mathcal{R}_\alpha\neg\varphi$ and $\mathfrak{M} \nvDash \mathcal{R}_\alpha\neg\psi$, so $\mathfrak{M} \nvDash \mathcal{R}_\alpha\neg\varphi \vee \mathcal{R}_\alpha\neg\psi$.

The specific axioms of the systems $R_{\mathfrak{K}}$, $R_{\mathfrak{P}}$, that is respectively (RA) and (RB), are the tautologies of the respective classes of models through the applicability of the conditions (13) (in the first case) and (14) (in the other). The specific axioms of the system $\mathsf{R}_{\mathfrak{C}}$—(1)—are tautologies through the conditions (13) i (14) taken together. **QED**

Theorem 9 (completeness) *Each of the systems* $\mathsf{R}_{\mathfrak{B}}, \mathsf{R}_{\mathfrak{K}}, \mathsf{R}_{\mathfrak{P}}, \mathsf{R}_{\mathfrak{C}}$ *is complete with respect to its set-theoretic semantics.*

Proof Due to 4, 5 and 6, and Tkaczyk's results concerning the adequate semantics of purely distributive systems CDEFGH, ACDEFGH, BCDEFGH, ABCDEFGH, it is sufficient to prove that any model determined by a Lindenbaum extension of a respective system, which is an extension of a purely distributive system, fulfills the conditions (54)–(57) corresponding to mutual interchange rules.

Let Λ be a Lindenbaum extension of any of the systems $\mathsf{R}_{\mathfrak{B}}$, $\mathsf{R}_{\mathfrak{K}}$, $\mathsf{R}_{\mathfrak{P}}$, $\mathsf{R}_{\mathfrak{C}}$. Assume that, for all $\alpha \in \mathbb{IN}, \varphi \in \mathbb{QF}$, $\mathfrak{d}(\alpha) \in \mathfrak{f}(\varphi)$ if and only if $R_\alpha\varphi \in \Lambda$. If $\varphi \in \mathbb{PL}$, $\mathfrak{f}(\varphi)$ is given by the definition of Lindenbaum extension (i.e. a maximal consistent set). Assume that the theorem holds for a subset Λ^* of Λ.

– According to (36)–(39), for every indicator α and quasi-formula φ, $\mathcal{R}_\alpha\varphi \in \Lambda^*$ iff $\mathcal{R}_\alpha\varphi^* \in \Lambda$, where φ^* is the quasi-formula resulting from application of the mutual interchange rule to quasi-formula φ. Thus for any indicator α, $\mathfrak{d}(\alpha) \in \mathfrak{f}(\varphi)$ iff $\mathfrak{d}(\alpha) \in \mathfrak{f}(\varphi^*)$, so $\mathfrak{f}(\varphi) = \mathfrak{f}(\varphi^*)$ and conditions (54)–(57) are satisfied.

For each formula A, if A is not a thesis of any of the systems under consideration, then according to the Lindenbaum theorem, there exists a complete and consistent extension of the system, such that A does not belong to that extension, thus there exists a countermodel for A. That in turn is equivalent to the proven theorem. **QED**

Thus all matrix systems $\mathsf{R}_{\mathfrak{C}}$, $\mathsf{R}_{\mathfrak{B}}$, $\mathsf{R}_{\mathfrak{K}}$ and $\mathsf{R}_{\mathfrak{P}}$ have adequate Tkaczyk-style set-theoretic semantics. It seems that an analogous result can be obtained for any system between the systems Zero and A-Q.

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SET-THEORETIC SEMANTICS
FOR MANY-VALUED POSITIONAL CALCULI

Summary

The objective of this paper is to formulate adequate set theoretic semantics for Tkaczyk's positional calculi $R_{\mathfrak{B}}$, $R_{\mathfrak{K}}$, and $R_{\mathfrak{P}}$ (TKACZYK 2007).

Keywords: positional calculi; matrix; set-thoretic semantics.

SEMANTYKA TEORIOMONOGOŚCIOWA
DLA WIELOWARTOŚCIOWYCH RACHUNKÓW POZYCYJNYCH

Streszczenie

Celem artykułu jest zdefiniowanie adekwatnych semantyk teoriomonogościowych dla rachunków pozycyjnych $R_{\mathfrak{B}}$, $R_{\mathfrak{K}}$, and $R_{\mathfrak{P}}$ (TKACZYK 2007).

Słowa kluczowe: rachunki pozycyjne; macierz; semantyka teoriomnogościowa.